CHSH Game

Using Convex Geometry

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1 CHSH Game

1.1 Background

The CHSH inequality, named after John Clauser, Michael Horne, Abner Shimony, and Richard Holt, provides support for Bell's Theorem, that quantum physics is incompatible with local hidden variable theories, experimentally.

The CHSH game involves two hypothetical experimentalists, Alice and Bob, who can not communicate. They are each given a random binary bit, Alice receives $x = 0, 1$ and Bob receives $y = 0, 1$. Their goal is to provide outputs $a = 0, 1$ and $b = 0, 1$ respectively, that satisfy the relation $xy = a \oplus b$.

1.2 Classical Physics

The most successful Alice and Bob can be at this game using classical physics only, is 75%, and they can achieve this by choosing outputs such that $a = b$, regardless of their input.

Classical strategies can be deterministic or randomized. For a deterministic strategy, Alice's output a must be a function of her random input x. Therefore, Alice must choose $a(x) = 0$, $a(x) = 1$, $a(x) = x$ or $a(x) = \neg x$. Similarly, Bob must choose $b(y) = 0$, $b(y) = 1$, $b(y) = y$ or $b(y) = \neg y$

x	y	$x \cdot y$	$a \oplus b$	Result
0	0	0	0	Win
0	1	0	0	Win
1	0	0	0	Win
1	1	1	0	Lose

Table 1: Win rate of 3/4 when a=b

	$a = 0$	$a = 1$	$a = x$	$a = \neg x$
$b = 0$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}$
$b = 1$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{4}$
$b = y$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$
$b = \neg y$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{4}$

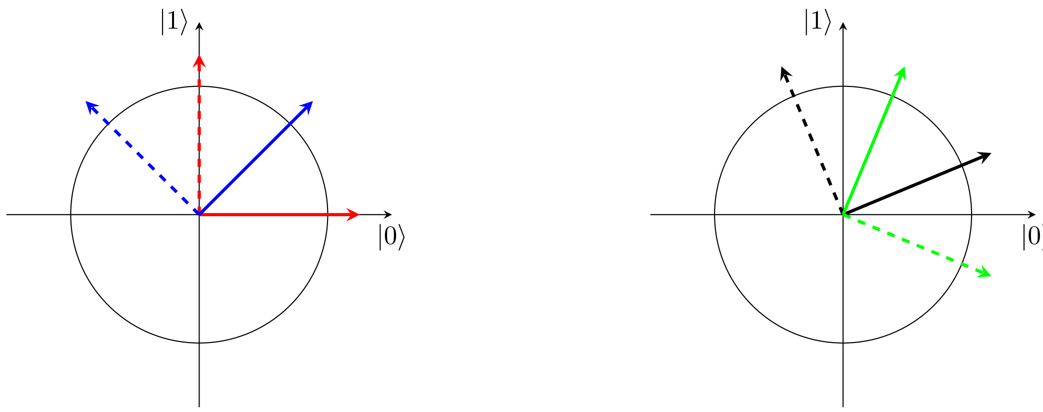
Table 2: Win Rates of all Deterministic Classical Strategies

Using the fact that any randomized strategy is a probability distribution of deterministic strategies, the maximum win rate for any randomized strategy is also 75%.

1.3 Quantum Physics

Using quantum mechanics, Alice and Bob can beat the classical strategy. They can do this by making use of quantum entanglement, by sharing a qubit each of an EPR pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. By exploiting correlations of the entangled EPR pair, Alice and Bob can win the game with a probability exceeding 75%.

They can do this by choosing their individual bases of measurement, dependent on the random input bit that they receive. If Alice receives $x = 1$, then she will measure a in the standard basis $\{|0\rangle, |1\rangle\}$, if she receives $x = 0$, then she will measure a in the basis $\{|+\rangle, |-\rangle\}$, which is a $\frac{\pi}{4}$ rotation from the standard basis $\{|0\rangle, |1\rangle\}$. Similarly, if Bob receives $y = 0$, he will then measure b in a basis rotated $\frac{\pi}{8}$ radians from the standard basis, and if he receives $y = 1$, he will then measure b in a basis rotated $-\frac{\pi}{8}$ radians from the standard basis. Output values for a and b correspond to the solid lines for $a, b = 0$ and to the dotted lines for $a, b = 1$.



(a) Red, x=1: Standard Basis
Blue, x=0: Rotated $\frac{\pi}{4}$ radians

(b) Black, y=0: Rotated $\frac{\pi}{8}$ radians
Green, y=1: Rotated $-\frac{\pi}{8}$ radians

Figure 1: Left: Alice's Bases **Right:** Bob's Bases

After Alice measures her qubit, Bobs qubit 'collapses' to whatever Alice has measured. For example, if Alice receives $x = 0$ as her input, she measures in the $\frac{\pi}{4}$ rotated basis, if she then measures $|+\rangle$, Bobs qubit 'snaps' to the solid blue line above, and he is $22.5°$ away from the outcome he requires for success in either of his measuring bases.

Using the fact that $\textbf{Prob}(\textbf{Win for a} = \textbf{b}) = \cos^{\mathbf{2}}(\theta - \gamma)$ and $\textbf{Prob}(\textbf{Win for a} \neq \textbf{b}) = \sin^{\mathbf{2}}(\theta - \gamma)$, where θ is the angle between Alice's measuring basis and the standard basis, and γ is the angle between Bob's measuring basis and the standard basis, we can produce the following table showing success probabilities. *Note:* we take the sine rather than cosine for the case where $x = y = 1$ as we want $a \neq b$ here.

x	y	θ	γ	$(\theta-\gamma)$	Prob(Win)
0	0	0	$\frac{\pi}{8}$	$-\frac{\pi}{8}$	$\cos^2(\frac{\pi}{8}) \approx 0.85$
0	1	0	$-\frac{\pi}{8}$	$\frac{\pi}{8}$	$\cos^2(\frac{\pi}{8}) \approx 0.85$
1	0	$\frac{\pi}{4}$	$\frac{\pi}{8}$	$\frac{\pi}{8}$	$\cos^2(\frac{\pi}{8}) \approx 0.85$
1	1	$\frac{\pi}{4}$	$-\frac{\pi}{8}$	$\frac{3\pi}{8}$	$\sin^2(\frac{3\pi}{8}) \approx 0.85$

Table 3: Quantum Strategy Achieves Success with Probability around 85%

2 Convex Polytopes

2.1 Background

A convex polytope is the convex hull of some finite set of points. A given polytope can be defined in two ways, either by its vertex representation or by its half-space representation.

A square is a simple example of a convex polytope. An example square may have a vertex representation of {(0,0), (2,0), (0,2), (2,2)}, its half-space representation can then be given by the following list of inequalities: $\{0 \leq x,\ 0 \leq 2-x,\ 0 \leq y,\ 0 \leq 2-y\}$

Although converting from the vertex representation to the half-space representation is simple for a 2-dimensional square, it is much more difficult for higher dimension polytopes. The execution of higher dimension half-space enumeration for this project was completed by 'pycddlib', which is a python wrapper for Komei Fukuda's cddlib. The programme is an implementation of the Double Description Method by Motzkin et al.

2.2 Local CHSH Polytope

In order to find the vertices for the local CHSH polytope, each of the conditional probabilities for Alice and Bob's potential inputs and outputs, $P(ab|xy) = P_A(a|x) \cdot P_B(b|y)$ for all $\{a, b, x, y\} \in \{0, 1\}$, were investigated. This gave $2^4 = 16$ possible vertices which are summarized in the following vector.

$$\begin{bmatrix} P(00|00) \\ P(00|01) \\ P(00|10) \\ P(00|11) \\ P(01|00) \\ P(01|01) \\ P(01|10) \\ P(01|11) \\ P(10|00) \\ P(10|01) \\ P(10|10) \\ P(10|11) \\ P(11|00) \\ P(11|01) \\ P(11|10) \\ P(11|11) \end{bmatrix} = \begin{bmatrix} P_A(0|0) \cdot P_B(0|0) \\ P_A(0|0) \cdot P_B(0|1) \\ P_A(0|1) \cdot P_B(0|0) \\ P_A(0|1) \cdot P_B(0|1) \\ P_A(0|0) \cdot P_B(1|0) \\ P_A(0|0) \cdot P_B(1|1) \\ P_A(0|1) \cdot P_B(1|0) \\ P_A(0|1) \cdot P_B(1|1) \\ P_A(1|0) \cdot P_B(0|0) \\ P_A(1|0) \cdot P_B(0|1) \\ P_A(1|1) \cdot P_B(0|0) \\ P_A(1|1) \cdot P_B(0|1) \\ P_A(1|0) \cdot P_B(1|0) \\ P_A(1|0) \cdot P_B(1|1) \\ P_A(1|1) \cdot P_B(1|0) \\ P_A(1|1) \cdot P_B(1|1) \end{bmatrix}$$

To find values for the conditional probabilities, we must look at all the potential scenarios. An example of which would be to take $P_A(0|0) = 0$, $P_A(0|1) = 0$, $P_B(0|0) = 0$ and $P_B(0|1) = 0$. *Note:* $P_A(0|0) = 0$ implies $P_A(1|0) = 1$ etc.

Evaluating all 16 choices for a, b, x and y produced the vertices for the CHSH polytope. As stated above, 'pycddlib' was used to perform half-space enumeration on the vertices. The code returned 16 vectors that corresponded to inequalities of the form $0 \leq \theta + x_1 + x_2 + ... + x_{16}$, where $\theta \in \mathbb{Z}$.

Translating the inequalities back to the conditional probabilities, retrieved the famous Bell Inequality,

$$\begin{gathered} P(00|00) - P(01|00) - P(10|00) + P(11|00) \\ +P(00|01) - P(01|01) - P(10|01) + P(11|01) \\ +P(00|10) - P(01|10) - P(10|10) + P(11|10) \\ -P(00|11) + P(01|11) + P(10|11) - P(11|11) \leq 2 \end{gathered}$$

2.3 Quantum CHSH Polytope

Using the mathematical formalisms of quantum mechanics, the maximum value for the Bell inequality given previously is

$$
\begin{gathered}
P(00|00) - P(01|00) - P(10|00) + P(11|00) \\
+P(00|01) - P(01|01) - P(10|01) + P(11|01) \\
+P(00|10) - P(01|10) - P(10|10) + P(11|10) \\
-P(00|11) + P(01|11) + P(10|11) - P(11|11) \leq 2\sqrt{2} > 2
\end{gathered}
$$

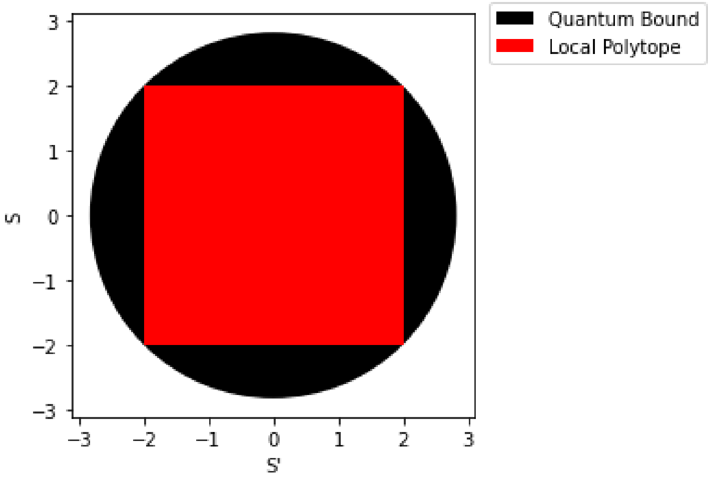


Figure 2: Quantum Violation of Bell CHSH Polytope

As can be seen from the above figure, and from our analysis of the CHSH game, quantum mechanics allows us to violate Bell inequalities. And the violation of these Bell inequalities provide near definitive demonstration that quantum physics cannot be represented by any version of the classical understanding of physics, including those of local hidden variable theories.