Some recent problems from the International Mathematical Olympiad

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IMO 2023 Problem 1

Determine all composite integers $n > 1$ that satisfy the following property: if $d_1, d_2, \dots, d_k$ are all the positive divisors of n, with $1 = d_1 < d_2 < \cdots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \leq i \leq k - 2$.

(Note: "d_i divides $d_{i+1} + d_{i+2}$" means that $d_{i+1} + d_{i+2}$ is a multiple of d_i.)

IMO 2022 Problem 1

The Bank of Oslo issues two types of coin: aluminium (denoted A) and bronze (denoted B). Marianne has n aluminium coins and n bronze coins, arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leq 2n$, Marianne repeatedly performs the following operation: she identifies the longest chain containing the kth coin from the left, and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be

$$AABBBABA \to BBBAAABA \to AAABBBBA \to BBBBAAAA \to BBBBAAAA \to \ldots$$

Find all pairs (n, k) with $1 \leq k \leq 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.

IMO 2021 Problem 1

Let $n \geq 100$ be an integer. Ivan writes the numbers $n, n+1, \ldots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

IMO 2020 Problem 4

There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points. The same conditions hold for B. We say that two stations are *linked* by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed).

Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

2019 Problem 1

Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a + b)).$$